

NEW TYPE INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS WITH APPLICATIONS

KHALED MEHREZ AND PRAVEEN AGARWAL

ABSTRACT. In this paper, we establish (presumably new type) integral inequalities for convex functions via the Hermite–Hadamard’s inequalities. As applications, we apply these new inequalities to construct inequalities involving special means of real numbers, some error estimates for the formula midpoint are given. Finally, new inequalities for some special and q –special functions are also pointed out.

Keywords: Hermite–Hadamard inequality, Integral inequalities, Convex functions, Special means, Midpoint formula, modified Bessel function, q –digamma function.

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1. INTRODUCTION

The theory of convexity is its close relationship with theory of inequalities. Many inequalities known in the literature are direct consequences of the applications of convex functions. An important inequality for convex functions which has been extensively studied in recent decades is Hermite–Hadamard’s inequality, which was obtained by Hermite and Hadamard independently. To be more precise, a function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a convex function, $a, b \in I$ with $a < b$, if and only if,

$$(1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

which is known as Hermite–Hadamard inequality. This inequalities (1) has become an important cornerstone in probability and optimization. An account on the history of this inequality can be found in [2]. The aim of this paper is to establish some new results connected with the Hermite–Hadamard inequalities (1). As application we derive some elementary inequalities for real numbers, some error estimates for the formula midpoint and we established new type inequalities for the modified Bessel functions of the first and second kind and the q –digamma function.

2. SOME PRELIMINARY LEMMAS

In this section, we state the following Lemmas, which are useful in the proofs of our main results.

Lemma 1. [1] Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on I^0 , and $a, b \in I$ with $a < b$, then we have

$$(2) \quad \frac{f(b)+f(a)}{2} - \frac{1}{b-a} \int_a^b f(x)dx = \frac{(b-a)^2}{2} \int_0^1 t(1-t)f''(ta+(1-t)b)dt.$$

Lemma 2. [3] Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on I^0 , and $a, b \in I$ with $a < b$, then we have

$$(3) \quad \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) = (b-a) \left[\int_0^{1/2} tf'(b+(a-b)t)dt + \int_{1/2}^1 (t-1)f'(b+(a-b)t)dt \right].$$

3. MAIN RESULTS

Theorem 1. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 with $a < b$. If f is a convex function, then the following inequalities holds:

$$(4) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{2f\left(\frac{a+b}{2}\right) + f\left(\frac{3b-a}{2}\right) + f\left(\frac{3a-b}{2}\right)}{4},$$

and

$$(5) \quad \left| \frac{1}{b-a} \int_a^b f(x)dx - \frac{f\left(\frac{a+b}{2}\right)}{2} \right| \leq \left| \frac{f\left(\frac{3b-a}{2}\right) + f\left(\frac{3a-b}{2}\right)}{4} \right|$$

Proof. By used the change of the variable $x = \frac{3}{4}t + \frac{a+b}{4}$, $t \in [\frac{3a-b}{3}, \frac{3b-a}{3}]$ we get

$$(6) \quad \begin{aligned} \frac{1}{b-a} \int_a^b f(x)dx &= \frac{3}{4(b-a)} \int_{\frac{3a-b}{3}}^{\frac{3b-a}{3}} f\left(\frac{3}{4}t + \frac{a+b}{4}\right)dx \\ &= \frac{3}{4(b-a)} \int_{\frac{3a-b}{3}}^{\frac{3b-a}{3}} f\left(\frac{1}{2}\left(\frac{3}{2}t\right) + \frac{1}{2}\left(\frac{a+b}{2}\right)\right)dx \\ &\leq \frac{3}{8(b-a)} \int_{\frac{3a-b}{3}}^{\frac{3b-a}{3}} \left[f\left(\frac{3}{2}t\right) + f\left(\frac{a+b}{2}\right) \right]dx \\ &= \frac{f\left(\frac{a+b}{2}\right)}{2} + \frac{1}{4(b-a)} \int_{\frac{3a-b}{2}}^{\frac{3b-a}{2}} f(t)dt. \end{aligned}$$

From the right hand side of inequality (1), we have

$$(7) \quad \frac{1}{4(b-a)} \int_{\frac{3a-b}{2}}^{\frac{3b-a}{2}} f(t)dt \leq \frac{f\left(\frac{3a-b}{2}\right) + f\left(\frac{3b-a}{2}\right)}{4}.$$

In view of (6) and (7), we deduce that the inequalities (4) and (5) holds true. ■

Theorem 2. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 with $a < b$, and its derivative $f' : [\frac{3a-b}{2}, \frac{3b-a}{2}] \rightarrow \mathbb{R}$, be a continuous on $[\frac{3a-b}{2}, \frac{3b-a}{2}]$. Let $q \geq 1$, if $|f'|^q$ is a convex function on $[\frac{3a-b}{2}, \frac{3b-a}{2}]$, then the following inequality holds

$$(8) \quad \left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} \left(\left| f'\left(\frac{3a-b}{2}\right) \right|^q + \left| f'\left(\frac{3b-a}{2}\right) \right|^q \right)^{1/q}.$$

Proof. By means of Lemma 2, we have

$$(9) \quad \frac{1}{2(b-a)} \int_{\frac{3a-b}{2}}^{\frac{3b-a}{2}} f(t)dt = f\left(\frac{a+b}{2}\right) + 2(b-a) \left[\int_0^{\frac{1}{2}} t f'\left(\frac{3b-a}{2} + 2(a-b)t\right)dt + \int_{\frac{1}{2}}^1 (t-1) f'\left(\frac{3b-a}{2} + 2(a-b)t\right)dt \right].$$

From (6) and (9), we get

$$(10) \quad \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \leq (b-a) \left[\int_0^{\frac{1}{2}} t \left| f'\left(\frac{3b-a}{2} + 2(a-b)t\right) \right| dt + \int_{\frac{1}{2}}^1 (1-t) \left| f'\left(\frac{3b-a}{2} + 2(a-b)t\right) \right| dt \right].$$

By the power-mean inequality, we find

$$\begin{aligned}
 (11) \quad \int_0^{1/2} t \left| f' \left(\frac{3b-a}{2} + 2(a-b)t \right) \right| dt &= \int_0^{1/2} t \left| f' \left(t \left(\frac{3a-b}{2} \right) + (1-t) \left(\frac{3b-a}{2} \right) \right) \right| dt \\
 &\leq \left(\int_0^{1/2} t dt \right)^{1-1/q} \left(\int_0^{1/2} t \left| f' \left(t \left(\frac{3a-b}{2} \right) + (1-t) \left(\frac{3b-a}{2} \right) \right) \right|^q dt \right)^{1/q} \\
 &\leq \left(\frac{1}{8} \right)^{1-1/q} \left(\int_0^{1/2} t^2 \left| f' \left(\frac{3a-b}{2} \right) \right|^q + t(1-t) \left| f' \left(\frac{3b-a}{2} \right) \right|^q \right)^{1/q} \\
 &= \left(\frac{1}{8} \right)^{1-1/q} \left(\frac{|f'(\frac{3a-b}{2})|^q}{24} + \frac{|f'(\frac{3b-a}{2})|^q}{12} \right)^{1/q}
 \end{aligned}$$

In the same way, we get

$$(12) \quad \int_{\frac{1}{2}}^1 (1-t) \left| f' \left(\frac{3b-a}{2} + 2(a-b)t \right) \right| dt \leq \left(\frac{1}{8} \right)^{1-1/q} \left(\frac{|f'(\frac{3a-b}{2})|^q}{12} + \frac{|f'(\frac{3b-a}{2})|^q}{24} \right)^{1/q}$$

So, (10), (11) and (12) completes the proof of this Theorem. ■

Theorem 3. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 with $a < b$, and its derivative $f' : [\frac{3a-b}{2}, \frac{3b-a}{2}] \rightarrow \mathbb{R}$, be a continuous on $[\frac{3a-b}{2}, \frac{3b-a}{2}]$. Let $q \geq 1$, if $|f'|^q$ is a convex function on $[\frac{3a-b}{2}, \frac{3b-a}{2}]$, then the following inequality holds

$$(13) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - f \left(\frac{a+b}{2} \right) \right| \leq (b-a) \left(\frac{1}{2^{p+1}(p+1)} \right)^{\frac{1}{p}} \left(\frac{|f'(\frac{3a-b}{2})|^q + |f'(\frac{3b-a}{2})|^q}{2} \right)^{1/q}.$$

Proof. By again the Hölder's inequality we have

$$\begin{aligned}
 (14) \quad \int_0^{1/2} t \left| f' \left(t \left(\frac{3a-b}{2} \right) + (1-t) \left(\frac{3b-a}{2} \right) \right) \right| dt &\leq \left[\int_0^{1/2} t^p dt \right]^{\frac{1}{p}} \left[\int_0^{1/2} \left| f' \left(t \left(\frac{3a-b}{2} \right) + (1-t) \left(\frac{3b-a}{2} \right) \right) \right|^q dt \right]^{\frac{1}{q}} \\
 &\leq \left[\frac{1}{(p+1)2^{p+1}} \right]^{\frac{1}{p}} \left[\left| f' \left(\frac{3a-b}{2} \right) \right|^q \int_0^{1/2} t dt + \left| f' \left(\frac{3b-a}{2} \right) \right|^q \int_0^{1/2} (1-t) dt \right]^{\frac{1}{q}} \\
 &= \left[\frac{1}{(p+1)2^{p+1}} \right]^{\frac{1}{p}} \left[\frac{|f'(\frac{3a-b}{2})|^q}{8} + 3 \frac{|f'(\frac{3b-a}{2})|^q}{8} \right]^{\frac{1}{q}}.
 \end{aligned}$$

Similarly, we get

(15)

$$\begin{aligned} \int_{1/2}^1 (1-t) \left| f' \left(t \left(\frac{3a-b}{2} \right) + (1-t) \left(\frac{3b-a}{2} \right) \right) \right| dt &\leq \left[\int_0^{1/2} t^p dt \right]^{\frac{1}{p}} \left[\int_{1/2}^1 \left| f' \left(t \left(\frac{3a-b}{2} \right) + (1-t) \left(\frac{3b-a}{2} \right) \right) \right|^q dt \right]^{\frac{1}{q}} \\ &\leq \left[\frac{1}{(p+1)2^{p+1}} \right]^{\frac{1}{p}} \left[\left| f' \left(\frac{3a-b}{2} \right) \right|^q \int_{1/2}^1 t dt + \left| f' \left(\frac{3b-a}{2} \right) \right|^q \int_{1/2}^1 (1-t) dt \right]^{\frac{1}{q}} \\ &= \left[\frac{1}{(p+1)2^{p+1}} \right]^{\frac{1}{p}} \left[\frac{3 \left| f' \left(\frac{3a-b}{2} \right) \right|^q + \left| f' \left(\frac{3b-a}{2} \right) \right|^q}{8} \right]^{\frac{1}{q}}. \end{aligned}$$

Thus, the inequalities (10), (14) and (15) completes the proof of this Theorem. \blacksquare

Corollary 1. *From Theorem 2-3 we get the following inequality for $q > 1$ we get*

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - f \left(\frac{a+b}{2} \right) \right| \leq \min\{K_1, K_2\} (b-a) \left(\left| f' \left(\frac{3a-b}{2} \right) \right|^q + \left| f' \left(\frac{3b-a}{2} \right) \right|^q \right)^{1/q}$$

where $K_1 = \frac{1}{8}$ and $K_2 = \left(\frac{1}{(p+1)2^{p+1+\frac{1}{pq}}} \right)^{\frac{1}{p}}$, such that $p = \frac{q}{q-1}$.

Theorem 4. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 with $a < b$, and its derivative $f' : [\frac{3a-b}{2}, \frac{3b-a}{2}] \rightarrow \mathbb{R}$, be a continuous on $[\frac{3a-b}{2}, \frac{3b-a}{2}]$. Let $q \geq 1$, if $|f''|^q$ is a convex function on $[\frac{3a-b}{2}, \frac{3b-a}{2}]$, then the following inequality holds*

(16)

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f \left(\frac{3a-b}{2} \right) + f \left(\frac{3b-a}{2} \right) + 2f \left(\frac{a+b}{2} \right)}{4} \right| \leq \frac{(b-a)^2}{3} \left[\frac{|f'' \left(\frac{3a-b}{2} \right)|^q + |f'' \left(\frac{3b-a}{2} \right)|^q}{2} \right]^{\frac{1}{q}}$$

Proof. From Lemma 1, we have

(17)

$$\frac{1}{2(b-a)} \int_{\frac{3a-b}{2}}^{\frac{3b-a}{2}} f(t) dt = \frac{f \left(\frac{3b-a}{2} \right) + f \left(\frac{3a-b}{2} \right)}{2} - 2(b-a)^2 \int_0^1 t(1-t) f'' \left(t \left(\frac{3a-b}{2} \right) + (1-t) \left(\frac{3b-a}{2} \right) \right) dt.$$

Thus, by (6) and (17) we obtain

(18)

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f \left(\frac{3a-b}{2} \right) + f \left(\frac{3b-a}{2} \right) + 2f \left(\frac{a+b}{2} \right)}{4} \right| \leq 2(b-a)^2 \int_0^1 t(1-t) \left| f'' \left(t \left(\frac{3a-b}{2} \right) + (1-t) \left(\frac{3b-a}{2} \right) \right) \right| dt.$$

Suppose that $q > 1$, from the Hölder's inequality for $q, p = \frac{q}{q-1}$ we get

$$\int_0^1 (t-t^2) \left| f'' \left(t \left(\frac{3a-b}{2} \right) + (1-t) \left(\frac{3b-a}{2} \right) \right) \right| dt =$$

$$\begin{aligned}
&= \int_0^1 [(t-t^2)]^{1-\frac{1}{q}} [(t-t^2)]^{\frac{1}{q}} \left| f'' \left(t \left(\frac{3a-b}{2} \right) + (1-t) \left(\frac{3b-a}{2} \right) \right) \right| dt \\
&\leq \left[\int_0^1 (t-t^2) \right]^{\frac{q-1}{q}} \left[\int_0^1 (t-t^2) \left| f'' \left(t \left(\frac{3a-b}{2} \right) + (1-t) \left(\frac{3b-a}{2} \right) \right) \right|^q dt \right]^{\frac{1}{q}} \\
&\leq \left(\frac{1}{6} \right)^{\frac{q-1}{q}} \left[\int_0^1 (t^2-t^3) \left| f'' \left(\frac{3a-b}{2} \right) \right|^q dt + \int_0^1 (t-t^2)(1-t) \left| f'' \left(\frac{3b-a}{2} \right) \right|^q dt \right]^{\frac{1}{q}} \\
&\leq \left(\frac{1}{6} \right)^{\frac{q-1}{q}} \left[\frac{|f''(\frac{3a-b}{2})|^q + |f''(\frac{3b-a}{2})|^q}{12} \right]^{\frac{1}{q}}.
\end{aligned}$$

Now suppose that $q = 1$, we obtain

$$\begin{aligned}
\int_0^1 (t-t^2) \left| f'' \left(t \left(\frac{3a-b}{2} \right) + (1-t) \left(\frac{3b-a}{2} \right) \right) \right| dt &\leq \int_0^1 (t-t^2) \left[t \left| f'' \left(\frac{3a-b}{2} \right) \right| + (1-t) \left| f'' \left(\frac{3b-a}{2} \right) \right| \right] dt \\
&= \frac{|f''(\frac{3a-b}{2})| + |f''(\frac{3b-a}{2})|}{12},
\end{aligned}$$

which completes the proof. ■

Remark 1. If $|f''(x)| \leq K$ on $[\frac{3a-b}{2}, \frac{3b-a}{2}]$ in Theorem 4, we get

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(\frac{3a-b}{2}) + f(\frac{3b-a}{2}) + 2f(\frac{a+b}{2})}{4} \right| \leq \frac{K(b-a)^2}{3}.$$

The other type is given by the next Theorem. For this, we note that the Beta function and gamma function are defined by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0, \quad \text{and} \quad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0.$$

The Beta function satisfied the following properties:

$$B(x, x) = 2^{1-2x} B(1/2, x), \quad \text{and} \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

In particular, we have

$$B(p+1, p+1) = 2^{1-2(p+1)} B(1/2, p+1) = 2^{1-2(p+1)} \frac{\Gamma(1/2)\Gamma(p+1)}{\Gamma(p+3/2)} = 2^{1-2(p+1)} \frac{\sqrt{\pi}\Gamma(p+1)}{\Gamma(p+3/2)}.$$

Theorem 5. Let $f : I^0 \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 with $a < b$, and its derivative $f' : [\frac{3a-b}{2}, \frac{3b-a}{2}] \rightarrow \mathbb{R}$, be a continuous on $[\frac{3a-b}{2}, \frac{3b-a}{2}]$. Let $q \geq 1$, if $|f''|^q$ is a convex function on $[\frac{3a-b}{2}, \frac{3b-a}{2}]$, then the following inequality holds

(19)

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(\frac{3a-b}{2}) + f(\frac{3b-a}{2}) + 2f(\frac{a+b}{2})}{4} \right| \leq \frac{(b-a)^2}{2} \left(\frac{\sqrt{\pi}\Gamma(p+1)}{2\Gamma(p+\frac{3}{2})} \right)^{\frac{1}{p}} \left(\frac{|f''(\frac{3a-b}{2})|^q + |f''(\frac{3b-a}{2})|^q}{2} \right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Again, by the Hölder's inequality and using the fact that the function $|f''|^q$ is convex we have

$$\int_0^1 (t-t^2) \left| f'' \left(t \left(\frac{3a-b}{2} \right) + (1-t) \left(\frac{3b-a}{2} \right) \right) \right| dt$$

$$\begin{aligned}
&\leq \left[\int_0^1 (t-t^2)^p \right]^{\frac{1}{p}} \left[\int_0^1 \left| f'' \left(t \left(\frac{3a-b}{2} \right) + (1-t) \left(\frac{3b-a}{2} \right) \right) \right|^q dt \right]^{\frac{1}{q}} \\
(20) \quad &\leq \left[\int_0^1 (t-t^2)^p \right]^{\frac{1}{p}} \left[\left| f'' \left(\frac{3a-b}{2} \right) \right|^q \int_0^1 t dt + \left| f'' \left(\frac{3b-a}{2} \right) \right|^q \int_0^1 (1-t) dt \right]^{\frac{1}{q}} \\
&= \left(\frac{\sqrt{\pi} \Gamma(p+1)}{2^{1+2p} \Gamma(p+\frac{3}{2})} \right)^{\frac{1}{p}} \left(\frac{|f''(\frac{3a-b}{2})|^q + |f''(\frac{3b-a}{2})|^q}{2} \right)^{\frac{1}{q}}.
\end{aligned}$$

Finally, from (18) and (20) we obtain the desired result. \blacksquare

Remark 2. With the above assumptions given that $|f''| \leq K$ on $[\frac{3a-b}{2}, \frac{3b-a}{2}]$ we obtain

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(\frac{3a-b}{2}) + f(\frac{3b-a}{2}) + 2f(\frac{a+b}{2})}{4} \right| \leq \frac{K(b-a)^2}{2} \left(\frac{\sqrt{\pi} \Gamma(p+1)}{2 \Gamma(p+\frac{3}{2})} \right)^{\frac{1}{p}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Another Hermite–Hadamard type inequality for powers in terms of the second derivatives is obtained as following:

Theorem 6. With the assumptions of Theorem 5 we have the inequality:

$$(21) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f(\frac{3a-b}{2}) + f(\frac{3b-a}{2}) + 2f(\frac{a+b}{2})}{4} \right| \leq (b-a)^2 \left(\left| f'' \left(\frac{3a-b}{2} \right) \right|^q + (q+1) \left| f'' \left(\frac{3b-a}{2} \right) \right|^q \right)^{\frac{1}{q}},$$

where

$$K(p, q) = 2 \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{1}{(q+1)(q+2)} \right)^{\frac{1}{q}}.$$

Proof. From the Hölder's inequality we have

$$\begin{aligned}
&\int_0^1 (t-t^2) \left| f'' \left(t \left(\frac{3a-b}{2} \right) + (1-t) \left(\frac{3b-a}{2} \right) \right) \right| dt \\
(22) \quad &\leq \left[\int_0^1 t^p dt \right]^{\frac{1}{p}} \left[\int_0^1 (1-t)^q \left| f'' \left(t \left(\frac{3a-b}{2} \right) + (1-t) \left(\frac{3b-a}{2} \right) \right) \right|^q dt \right]^{\frac{1}{q}} \\
&\leq \left[\int_0^1 t^p dt \right]^{\frac{1}{p}} \left[\left| f'' \left(\frac{3a-b}{2} \right) \right|^q \int_0^1 t(1-t)^q dt + \left| f'' \left(\frac{3b-a}{2} \right) \right|^q \int_0^1 (1-t)^{q+1} dt \right]^{\frac{1}{q}} \\
&= \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[B(2, q+1) \left| f'' \left(\frac{3a-b}{2} \right) \right|^q + \frac{|f''(\frac{3b-a}{2})|^q}{q+2} \right]^{\frac{1}{q}} \\
&= \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{1}{(q+1)(q+2)} \right)^{\frac{1}{q}} \left(\left| f'' \left(\frac{3a-b}{2} \right) \right|^q + (q+1) \left| f'' \left(\frac{3b-a}{2} \right) \right|^q \right)^{\frac{1}{q}}.
\end{aligned}$$

So, the proof of Theorem 6 is completes. \blacksquare

A similar result is embodied in the following Theorem.

Theorem 7. *et $f : I^0 \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I^0 with $a < b$, and its derivative $f' : [\frac{3a-b}{2}, \frac{3b-a}{2}] \rightarrow \mathbb{R}$, be a continuous on $[\frac{3a-b}{2}, \frac{3b-a}{2}]$. Let $q \geq 1$, if $|f''|^q$ is a convex function on $[\frac{3a-b}{2}, \frac{3b-a}{2}]$, then the following inequality holds:*

$$(23) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f\left(\frac{3a-b}{2}\right) + f\left(\frac{3b-a}{2}\right) + 2f\left(\frac{a+b}{2}\right)}{4} \right| \leq (b-a)^2 \left(2 \left| f''\left(\frac{3a-b}{2}\right) \right|^q + (q+1) \left| f''\left(\frac{3b-a}{2}\right) \right|^q \right)^{\frac{1}{q}},$$

where

$$K_2(q) = \left(\frac{2}{(q+1)(q+2)(q+3)} \right)^{\frac{1}{q}}.$$

Proof. From the power-mean inequality we obtain

$$(24) \quad \begin{aligned} & \int_0^1 (t-t^2) \left| f''\left(t\left(\frac{3a-b}{2}\right) + (1-t)\left(\frac{3b-a}{2}\right)\right) \right| dt \\ & \leq \left[\int_0^1 t dt \right]^{1-\frac{1}{q}} \left[\int_0^1 t(1-t)^q \left| f''\left(t\left(\frac{3a-b}{2}\right) + (1-t)\left(\frac{3b-a}{2}\right)\right) \right|^q dt \right]^{\frac{1}{q}} \\ & \leq \left[\int_0^1 t dt \right]^{1-\frac{1}{q}} \left[\left| f''\left(\frac{3a-b}{2}\right) \right|^q \int_0^1 t^2(1-t)^q dt + \left| f''\left(\frac{3b-a}{2}\right) \right|^q \int_0^1 t(1-t)^{q+1} dt \right]^{\frac{1}{q}} \\ & = \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left[B(3, q+1) \left| f''\left(\frac{3a-b}{2}\right) \right|^q + B(2, q+2) \left| f''\left(\frac{3b-a}{2}\right) \right|^q \right]^{\frac{1}{q}} \\ & = \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left(\frac{1}{(q+1)(q+2)(q+3)} \right)^{\frac{1}{q}} \left(2 \left| f''\left(\frac{3a-b}{2}\right) \right|^q + (q+1) \left| f''\left(\frac{3b-a}{2}\right) \right|^q \right)^{\frac{1}{q}}. \end{aligned}$$

which completes the proof of Theorem 7. ■

Corollary 2. *From Theorem 5–7 we get the following inequality for $q > 1$*

$$\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{f\left(\frac{3a-b}{2}\right) + f\left(\frac{3b-a}{2}\right) + 2f\left(\frac{a+b}{2}\right)}{4} \right| \leq \min(K_3, K_4, K_5, K_6)(b-a)^2,$$

where

$$\begin{aligned} K_3 &= \frac{1}{3} \left(\frac{|f''\left(\frac{3a-b}{2}\right)|^q + |f''\left(\frac{3b-a}{2}\right)|^q}{2} \right)^{\frac{1}{q}} \\ K_4 &= 2 \left(\frac{\sqrt{\pi}\Gamma(p+1)}{2\Gamma(p+\frac{3}{2})} \right)^{\frac{1}{p}} \left(\frac{|f''\left(\frac{3a-b}{2}\right)|^q + |f''\left(\frac{3b-a}{2}\right)|^q}{2} \right)^{\frac{1}{q}} \\ K_5 &= 2 \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{1}{(q+1)(q+2)} \right)^{\frac{1}{q}} \left(\left| f''\left(\frac{3a-b}{2}\right) \right|^q + (q+1) \left| f''\left(\frac{3b-a}{2}\right) \right|^q \right)^{\frac{1}{q}} \\ K_6 &= \left(\frac{2}{(q+1)(q+2)(q+3)} \right)^{\frac{1}{q}} \left(2 \left| f''\left(\frac{3a-b}{2}\right) \right|^q + (q+1) \left| f''\left(\frac{3b-a}{2}\right) \right|^q \right)^{\frac{1}{q}}. \end{aligned}$$

4. APPLICATIONS

4.1. Applications to special means. Now using the results of Section 3, we give some applications to special means of positive real numbers.

1. The arithmetic mean:

$$A = A(a, b) = \frac{a+b}{2}; \quad a, b \in \mathbb{R}, \quad \text{with } a, b > 0.$$

2. The geometric mean:

$$G = G(a, b) = \sqrt{ab}; \quad a, b \in \mathbb{R}, \quad \text{with } a, b > 0.$$

3. The logarithmic mean:

$$L(a, b) = \frac{b-a}{\log b - \log a}.$$

4. The generalized logarithmic mean:

$$L_n(a, b) = \left[\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)} \right]^{\frac{1}{n}}; \quad n \in \mathbb{Z} \setminus \{-1, 0\}, \quad a, b \in \mathbb{R}, \quad \text{with } a, b > 0.$$

Proposition 1. *Let $n \in \mathbb{Z} \setminus \{-1, 0\}$ and $a, b \in \mathbb{R}$ such that $0 < a < b$. Then the following inequalities*

$$(25) \quad |2L_n^n(a, b) - A^n(a, b)| \leq A \left(\left| \frac{3b-a}{2} \right|^n, \left| \frac{3a-b}{2} \right|^n \right),$$

and

$$(26) \quad |A^n(a, b) - L_n^n(a, b)| \leq \min\{K_1, K_2\} (2^{\frac{1}{n}} |n| (b-a)) \left[A \left(\left| \frac{3a-b}{2} \right|^{(n-1)q}, \left| \frac{3b-a}{2} \right|^{(n-1)q} \right) \right]^{1/q},$$

holds.

Proof. The assertion follows from Theorem 1 and Theorem 2–3 for $f(x) = x^n$ and n as specified above. ■

Proposition 2. *Let $a, b \in \mathbb{R}$ such that $0 < a < b$. Then the following inequalities*

$$(27) \quad |2G^{-2}(a, b) - A^{-2}(a, b)| \leq A \left(\left(\frac{3a-b}{2} \right)^{-2}, \left(\frac{3b-a}{2} \right)^{-2} \right),$$

and

$$(28) \quad |G^{-2}(a, b) - A^{-2}(a, b)| \leq \min\{K_1, K_2\} (4^{\frac{1}{q}} (b-a)) \left[A \left(\left| \frac{3a-b}{2} \right|^{-3q}, \left| \frac{3b-a}{2} \right|^{-3q} \right) \right]^{1/q},$$

are valid.

Proof. The assertion follows from Theorem 1 and Theorem 2–3 applied for $f(x) = \frac{1}{x^2}$. ■

Proposition 3. *Let $q \geq 1$ and $a, b \in \mathbb{R}$ such that $0 < a < b$. Then the following inequalities*

$$(29) \quad |A^{-1}(a, b) - 2L^{-1}(a, b)| \leq A \left(\left| \frac{3a-b}{2} \right|^{-1}, \left| \frac{3b-a}{2} \right|^{-1} \right),$$

and

$$(30) \quad |A^{-1}(a, b) - L^{-1}(a, b)| \leq \min\{K_1, K_2\}(2^{\frac{1}{q}}(b-a)) \left[A \left(\left| \frac{3a-b}{2} \right|^{-2q}, \left| \frac{3b-a}{2} \right|^{-2q} \right) \right]^{1/q},$$

holds true.

Proof. The assertion follows from Theorem 1 and Theorem 2-3 for $f(x) = \frac{1}{x}$. ■

4.2. The midpoint formula. Let d be a partition $a = x_0 < x_1 < \dots < x_{m-1} < x_m = b$ of the interval $[a, b]$ and consider the quadrature formula

$$\int_a^b f(x)dx = T_i(f, d) + E_i(f, d), \quad i = 1, 2,$$

where

$$T_1(f, d) = \sum_{i=0}^{m-1} \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i)$$

for the trapezoidal version and

$$T_2(f, d) = \sum_{i=0}^{m-1} f\left(\frac{x_i + x_{i+1}}{2}\right) (x_{i+1} - x_i)$$

for the midpoint version and $E_i(f, d)$ denotes the associated approximation error.

Proposition 4. Suppose that the function f is convex, then for every partition of $[a, b]$ the midpoint error satisfies

$$(31) \quad \left| \int_a^b f(x)dx + E_2(f, d) \right| \leq \sum_{i=0}^{m-1} (x_{i+1} - x_i) \frac{\left| f\left(\frac{3x_i - x_{i+1}}{2}\right) + f\left(\frac{3x_{i+1} - x_i}{2}\right) \right|}{2} \\ \leq \sum_{i=1}^{m-1} (x_{i+1} - x_i) \max \left(\left| f\left(\frac{3x_i - x_{i+1}}{2}\right) \right|, \left| f\left(\frac{3x_{i+1} - x_i}{2}\right) \right| \right).$$

Proof. From Theorem 1, we have

$$\left| 2 \int_{x_i}^{x_{i+1}} f(x)dx - (x_{i+1} - x_i) f\left(\frac{x_i + x_{i+1}}{2}\right) \right| \leq \left| \frac{f\left(\frac{3x_i - x_{i+1}}{2}\right) + f\left(\frac{3x_{i+1} - x_i}{2}\right)}{2} \right|.$$

On the other hand, we have

$$(32) \quad \left| \int_a^b f(x)dx + \left\{ \int_a^b f(x)dx - T_2(f, d) \right\} \right| = \left| \sum_{i=0}^{m-1} \left\{ 2 \int_{x_i}^{x_{i+1}} f(x)dx - (x_{i+1} - x_i) f\left(\frac{x_i + x_{i+1}}{2}\right) \right\} \right| \\ \leq \sum_{i=0}^{m-1} (x_{i+1} - x_i) \frac{\left| f\left(\frac{3x_i - x_{i+1}}{2}\right) + f\left(\frac{3x_{i+1} - x_i}{2}\right) \right|}{2} \\ \leq \sum_{i=1}^{m-1} (x_{i+1} - x_i) \max \left(\left| f\left(\frac{3x_i - x_{i+1}}{2}\right) \right|, \left| f\left(\frac{3x_{i+1} - x_i}{2}\right) \right| \right). \quad \blacksquare$$

Proposition 5. Suppose that the function $|f'|^q$, $q \leq 1$, then for every partition of $[a, b]$ the midpoint error satisfies

$$(33) \quad \begin{aligned} |E_2(f; d)| &\leq \min(K_1, K_2) \sum_{i=1}^{m-1} (x_{i+1} - x_i)^2 \left[\left| f' \left(\frac{3x_i - x_{i+1}}{2} \right) \right|^q + \left| f' \left(\frac{3x_{i+1} - x_i}{2} \right) \right|^q \right]^{\frac{1}{q}} \\ &\leq 2 \min(K_1, K_2) \sum_{i=1}^{m-1} (x_{i+1} - x_i)^2 \max \left(\left| f' \left(\frac{3x_i - x_{i+1}}{2} \right) \right|, \left| f' \left(\frac{3x_{i+1} - x_i}{2} \right) \right| \right). \end{aligned}$$

Proof. From Corollary 1, we obtain

$$\left| \int_{x_i}^{x_{i+1}} f(x) dx - (x_{i+1} - x_i) f \left(\frac{x_i + x_{i+1}}{2} \right) \right| \leq \min(K_1, K_2) (x_{i+1} - x_i)^2 \left[\left| f' \left(\frac{3x_i - x_{i+1}}{2} \right) \right|^q + \left| f' \left(\frac{3x_{i+1} - x_i}{2} \right) \right|^q \right]^{\frac{1}{q}}.$$

On the other hand, we have

$$(34) \quad \begin{aligned} \left| \int_a^b f(x) dx - T_2(f, d) \right| &= \left| \sum_{i=0}^{m-1} \left\{ \int_{x_i}^{x_{i+1}} f(x) dx - (x_{i+1} - x_i) f \left(\frac{x_i + x_{i+1}}{2} \right) \right\} \right| \\ &\leq \min(K_1, K_2) \sum_{i=0}^{m-1} (x_{i+1} - x_i)^2 \left[\left| f' \left(\frac{3x_i - x_{i+1}}{2} \right) \right|^q + \left| f' \left(\frac{3x_{i+1} - x_i}{2} \right) \right|^q \right]^{\frac{1}{q}} \\ &\leq 2 \min(K_1, K_2) \sum_{i=1}^{m-1} (x_{i+1} - x_i)^2 \max \left(\left| f' \left(\frac{3x_i - x_{i+1}}{2} \right) \right|, \left| f' \left(\frac{3x_{i+1} - x_i}{2} \right) \right| \right). \end{aligned}$$

■

4.3. Inequalities for some special functions.

4.3.1. *Modified Bessel functions.* Recall that the modified Bessel function I_p of the first kind has the series representation [[4], p. 77]

$$I_p(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{p+2n}}{n! \Gamma(p+n+1)},$$

where $x \in \mathbb{R}$, while the modified Bessel function of the second kind K_p [[4], p. 78] is usually defined as

$$K_p(x) = \frac{\pi}{2} \frac{I_{-p}(x) + I_p(x)}{\sin p\pi}.$$

For this we consider the function $\mathcal{I}_p : \mathbb{R} \rightarrow [1, \infty)$, defined by

$$\mathcal{I}_p(x) = 2^p \Gamma(p+1) x^{-p} I_p(x),$$

where $\Gamma(\cdot)$ is the gamma function.

Proposition 6. Let $p > -1$, $a, b \in \mathbb{R}$ such that $0 < a < b$, then the following inequality holds

$$(35) \quad \left| \frac{\mathcal{I}_p(b) - \mathcal{I}_p(a)}{b - a} \right| \leq \frac{\left[\left(\frac{3a-b}{2} \right) \mathcal{I}_{p+1} \left(\frac{3a-b}{2} \right) + \left(\frac{3b-a}{2} \right) \mathcal{I}_{p+1} \left(\frac{3b-a}{2} \right) \right] + (a+b) \mathcal{I}_{p+1} \left(\frac{a+b}{2} \right)}{8(p+1)}.$$

In particular, the following inequality

$$(36) \quad \left| \frac{\cosh(b) - \cosh(a)}{b - a} \right| \leq \frac{\sinh \left(\frac{3a-b}{2} \right) + \sinh \left(\frac{3b-a}{2} \right) + 2 \sinh \left(\frac{a+b}{2} \right)}{4}.$$

is true.

Proof. Observe that the function $x \mapsto \mathcal{I}'_p(x)$ is convex on $[0, \infty)$, since as power series has only positive coefficients. Now, from Theorem 1 we obtain

$$(37) \quad \left| \frac{\mathcal{I}_p(b) - \mathcal{I}_p(a)}{b - a} \right| \leq \frac{2\mathcal{I}'_p\left(\frac{a+b}{2}\right) + \mathcal{I}'_p\left(\frac{3a-b}{2}\right) + \mathcal{I}'_p\left(\frac{3b-a}{2}\right)}{4}.$$

By using the differentiation formula [[4], p. 79]

$$(38) \quad \mathcal{I}'_p(x) = \frac{x}{2(p+1)} \mathcal{I}_{p+1}(x)$$

we deduce that the inequality (35) holds. Now taking into account the relations $\mathcal{I}_{-1/2}(x) = \cosh(x)$ and $\mathcal{I}_{1/2}(x) = \sinh(x)/x$, the inequality (35) reduce to inequality (36). ■

Proposition 7. Let $a, b \in \mathbb{R}$ such that $0 < a < b$. Suppose that $3a \neq b$, then the following inequality holds true

$$(39) \quad \left| \frac{a^p K_p(b) - b^p K_p(a)}{(ab)^p(b-a)} \right| \leq \frac{F_p(a, b)}{[(a+b)(3a-b)(3b-a)]^p},$$

for all $p > 1$, where

$$F_p(a, b) = 2^{p+1}[(3a-b)(3b-a)]^p K_{p+1}\left(\frac{a+b}{2}\right) + [2(a+b)(3b-a)]^p K_{p+1}\left(\frac{3a-b}{2}\right) + [2(a+b)(3a-b)]^p K_{p+1}\left(\frac{3b-a}{2}\right).$$

Proof. Using the integral representation [4], p. 181

$$K_p(x) = \int_0^\infty e^{-x \cosh t} \cosh(pt) dt, \quad x > 0$$

where $p \in \mathbb{R}$, we conclude that the function $x \mapsto K_p(x)$ is completely monotonic on $(0, \infty)$ for each $p \in \mathbb{R}$. By the Leibniz formula for derivatives the product of two completely monotonic functions is completely monotonic, we conclude that the function $x \mapsto \frac{K_p(x)}{x^p}$ is strictly completely monotonic on $(0, \infty)$ for all $p > 1$. Now, let $f_p(x) = -(\frac{K_p(x)}{x^p})'$, so the function $f_p(x)$ is completely monotonic on $(0, \infty)$ for all $p > 1$, and consequently is convex. Using Theorem 1 and the fact that [[4], p. 79] $(\frac{K_p(x)}{x^p})' = -\frac{K_{p+1}(x)}{x^p}$, we conclude that the inequality (39) is holds for all $p > 1$. ■

4.4. q -digamma function. Let $0 < q < 1$, the q -digamma function ψ_q , is the q -analogue of the psi or digamma function ψ defined by

$$(40) \quad \begin{aligned} \psi_q(x) &= -\ln(1-q) + \ln q \sum_{k=0}^{\infty} \frac{q^{k+x}}{1-q^{k+x}} \\ &= -\ln(1-q) + \ln q \sum_{k=1}^{\infty} \frac{q^{kx}}{1-q^k}. \end{aligned}$$

For $q > 1$ and $x > 0$, the q -digamma function ψ_q is defined by

$$\begin{aligned} \psi_q(x) &= -\ln(q-1) + \ln q \left[x - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{q^{-(k+x)}}{1-q^{-(k+x)}} \right] \\ &= -\ln(q-1) + \ln q \left[x - \frac{1}{2} - \sum_{k=1}^{\infty} \frac{q^{-kx}}{1-q^{-kx}} \right]. \end{aligned}$$

Proposition 8. For $a, b \in \mathbb{R}$, such that $0 < a < b$. Then the following inequality

$$(41) \quad \left| \frac{\psi_q(b) - \psi_q(a)}{b - a} \right| \leq \frac{[\psi'_q(|\frac{3a-b}{2}|) + \psi'_q(\frac{3b-a}{2}) + 2\psi'_q(\frac{a+b}{2})]}{4}$$

holds true for all $q > 0$.

Proof. By using the definitions of the function $\psi_q(x)$ we deduce that the function $x \mapsto \psi'_q(x)$ is completely monotonic on $(0, \infty)$ for all $q > 0$, and consequently the function $x \mapsto \psi'_q(x)$ is convex on $(0, \infty)$. Now applying Theorem 1 we deduce that the inequality is valid for all $q > 0$. ■

Proposition 9. For $a, b \in \mathbb{R}$, such that $0 < a < b$. Then the following inequality

$$(42) \quad \left| \frac{\psi_q(b) - \psi_q(a)}{b - a} - \frac{[\psi'_q(|\frac{3a-b}{2}|) + \psi'_q(\frac{3b-a}{2}) + 2\psi'_q(\frac{a+b}{2})]}{4} \right| \leq \frac{(b-a)^2 [\psi_q^{(3)}(|\frac{3a-b}{2}|) + \psi_q^{(3)}(\frac{3b-a}{2})]}{6},$$

holds true for all $q > 0$.

Proof. We set $f = \psi'_q$, thus the function $f'' = \psi_q^{(3)}$ is completely monotonic on $(0, \infty)$ for all $q > 0$. Apply Theorem 4 we obtain the desired inequality. ■

Concluding Remarks: Lastly, we conclude this paper by remarking that, we have obtained new type integral inequalities for convex functions and some of their applications. All the inequalities are interesting and important in the field of integral inequalities.

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KHALED MEHREZ. DÉPARTEMENT DE MATHÉMATIQUES ISSAT KASSERINE, UNIVERSITÉ DE KAIROUAN, TUNISIA.
E-mail address: k.mehrez@yahoo.fr

PRAVEEN AGARWAL. DEPARTMENT OF MATHEMATICS, ANAND INTERNATIONAL COLLEGE OF ENGINEERING, JAIPUR, RAJASTHAN, INDIA
E-mail address: goyal.praveen@gmail.com